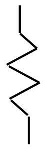
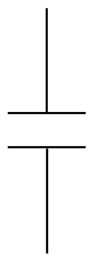
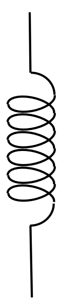
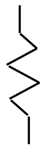
Inductive Reactance, Capacitive Reactance and Impedance

element	symbol	units	resistive nature	Filter?	phase relationship
Resistor					
Capacitor					
Inductor					
RLC ckt.	Z				

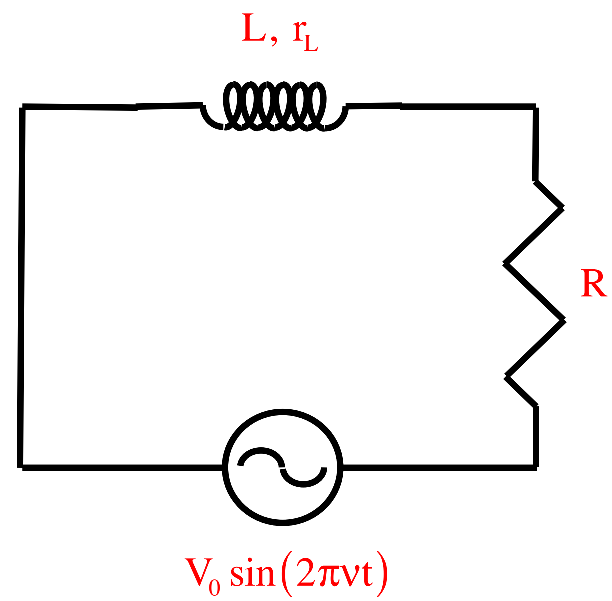
Summary of the table

- **Resistors:**
  - Resistors have a resistive nature that is characterized as "resistance" and that has the units of ohms.
  - The resistive nature (i.e., the resistance) of a resistor is not frequency dependent. That is, it doesn't matter at what frequency the AC power supply is acting, the resistive nature of the resistor will always be the same.
  - Because the resistive nature of the resistor is not frequency dependent, it doesn't act as a frequency filter in any frequency range.

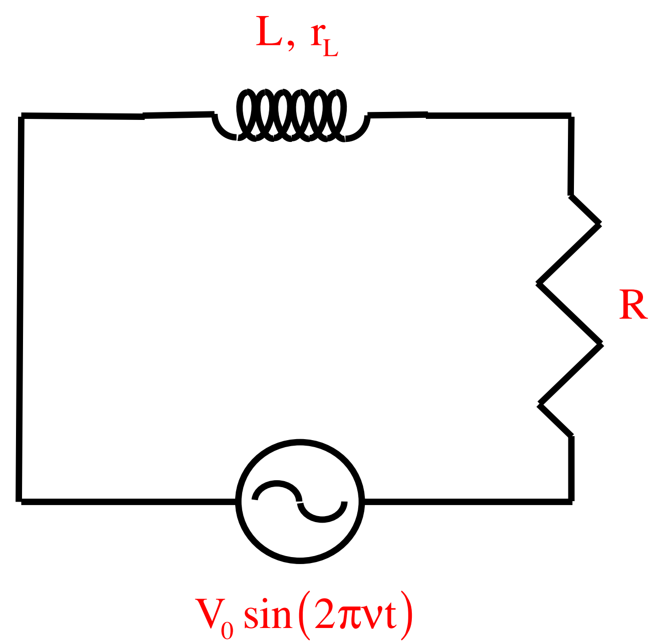
element	symbol	units	resistive nature	Filter?	phase relationship
Resistor	 R	ohms	"resistor-like" resistance R (ohms)	no	current in phase with voltage across element
Capacitor					
Inductor					
RLC ckt.					

Intro to Inductors

- **Inductors**:
  - Inductors do have a "resistor-like" resistance to them that is due to the resistance inherent within the wire making up the inductor's coil. That resistance is characterized as r_L. In addition, they also have a frequency-dependent resistive nature.
  - The inductor's frequency-dependent resistive nature is called the inductive reactance. As do all resistive natures, it has the units of ohms.
  - To see why an inductor has this frequency-dependent resistive nature, consider the RL circuit to the right.



The conceptional rationale for inductive reactance:



1.) Restatement of already established fact:

a.) the voltage across a power supply will equal the sum of the voltage drops across the various elements in the circuit.

b.) the elements in this circuit are an inductor and a resistor;

c.) the current in a circuit with a resistor will be proportional to the voltage across the resistor.

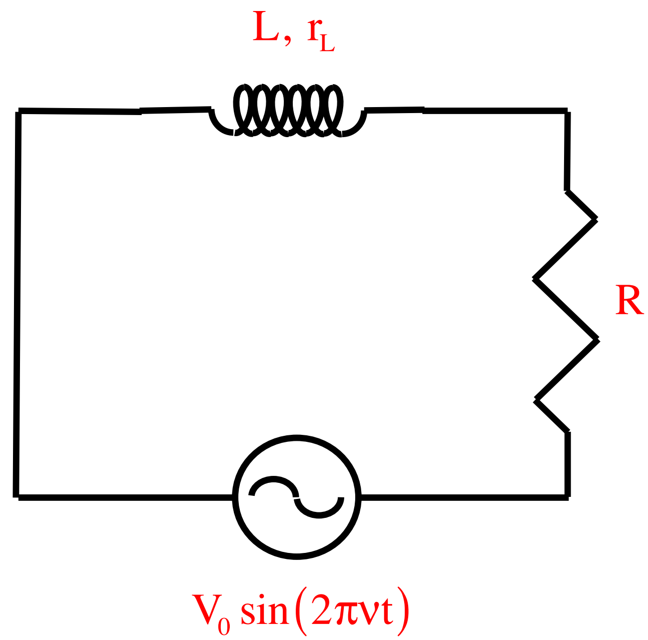
2.) At low frequency, the change of current with time will be relatively small so the induced EMF (voltage) will be small, which means the voltage across the resistor will be big. Soooo, at low frequency, there will be a big current and the inductor is NOT providing much resistance to charge flow (i.e., its inductive reactance will be small).

3.) It's exactly opposite at high frequency with the inductive reactance being large, so inductors are low pass filters.

Quantitatively, what is the inductor's resistive nature?

a.) Writing out Kirchoff's loop equation for this circuit, we get:

$$-i\left(R + r_L\right) - L\frac{di}{dt} + V_o \sin\left(2\pi\nu t\right) = 0$$



b.) The solution of this differential equation will be of the form:

$$i = \frac{\text{voltage term}}{\text{net resistive nature}}$$

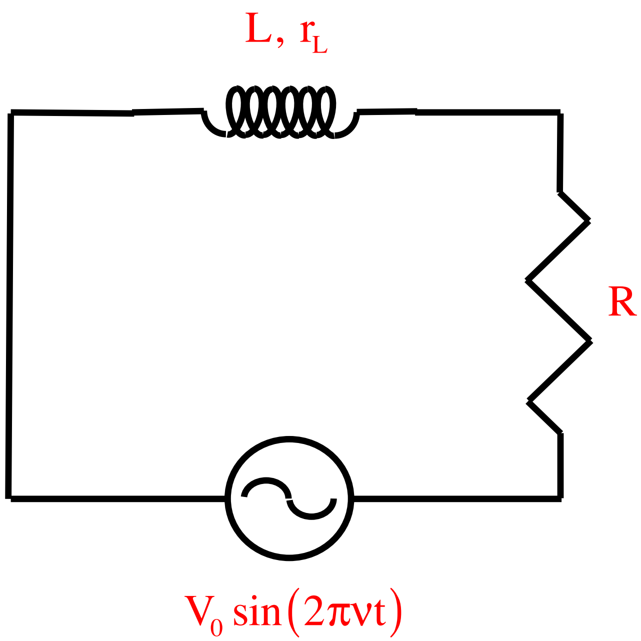
c.) If we did the math, part of that denominator would be a frequency-dependent quantity

$$2\pi\nu L \quad \text{(ohms)}$$

d.) This is the inductive reactance X_L, the frequency-dependent resistive nature of an inductor.

Lastly, the phase shift:

4.) For an inductor, the induced emf across it is found by $\varepsilon = -L\frac{di}{dt}$, so the current function should be the first derivative of the voltage function.

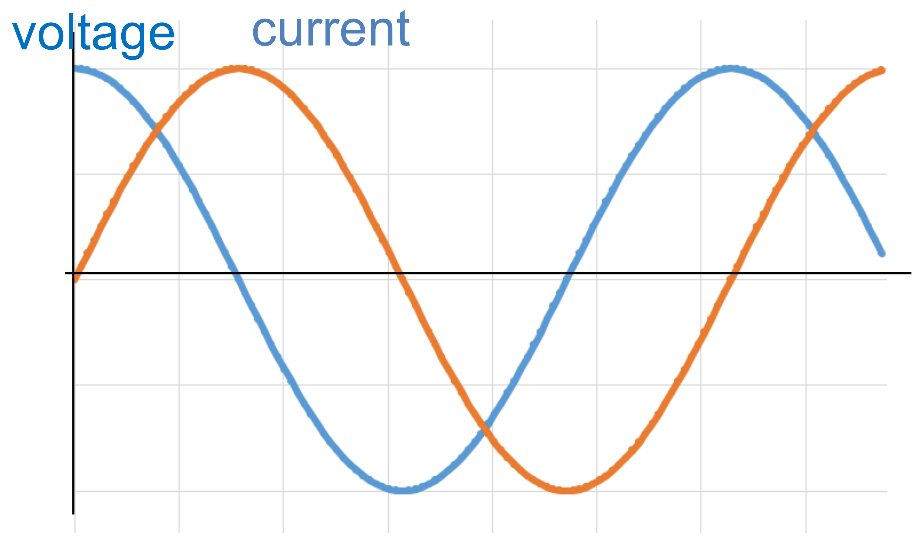


Knowing that…

The voltage across the power supply and the current through a circuit (which is proportional to the voltage across the resistor) will not be in phase with each other!

What's the deal with the phase shift?

1.) So if we take the derivative of the current function, we get the voltage function:

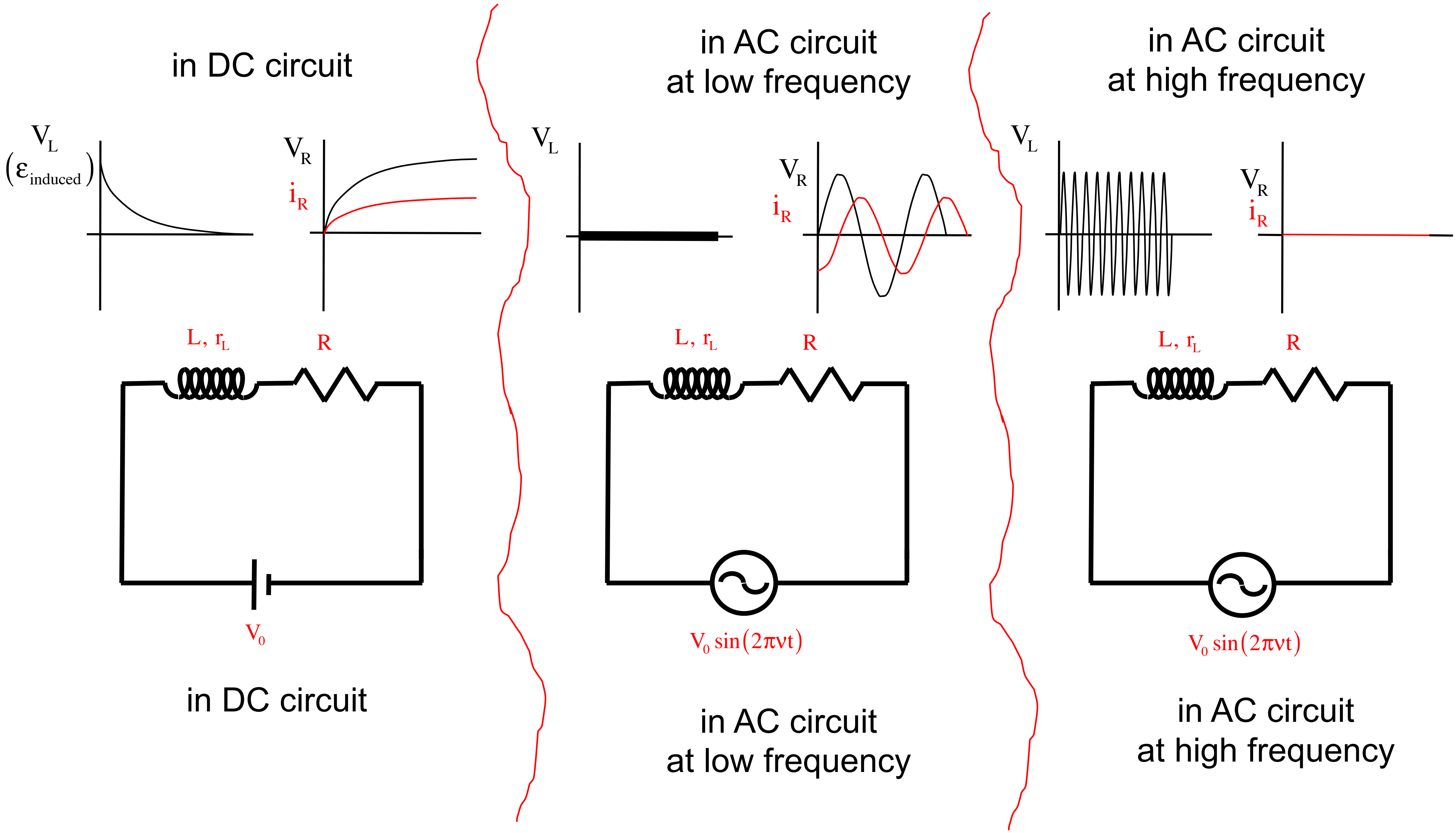


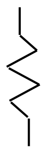
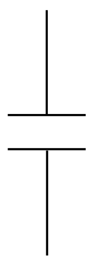
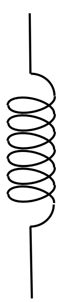
2.) What this shows is that the voltage function LEADS the current function, and for a circuit with minimal resistor-like resistance in it, that lead will be

$$\frac{\pi}{2} \text{ radians}$$

3.) Conceptually, this also makes sense: the inductor will fight current when its voltage changes the most (which is at the maximum), but allow current when its voltage change is minimal, so current will flow after peak voltage has been reached.

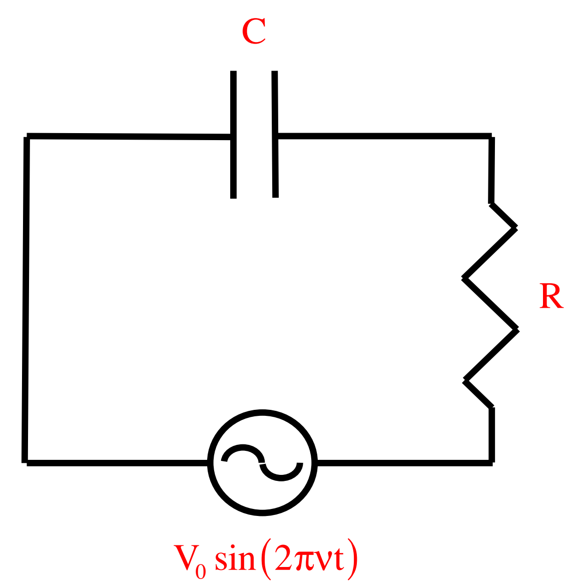
RL circuit:


in DC circuit
V_L
$(\varepsilon_{induced})$
V_R
i_R
L, r_L
R
V_0
in DC circuit
in AC circuit
at low frequency
V_L
V_R
i_R
L, r_L
R
$V_0 \sin(2\pi\nu t)$
in AC circuit
at low frequency
in AC circuit
at high frequency
V_L
V_R
i_R
L, r_L
R
$V_0 \sin(2\pi\nu t)$
in AC circuit
at high frequency

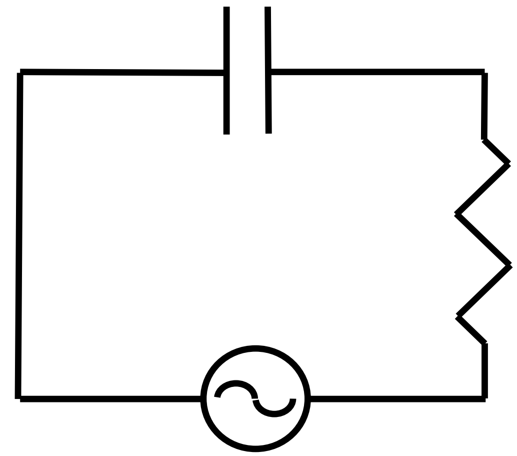
element	symbol	units	resistive nature	Filter?	phase relationship
Resistor	 R	ohms	"resistor-like" resistance R (ohms)	no	current in phase with voltage across element
Capacitor					
Inductor	 L	henrys	at low ν, V_L small, V_R big $\Rightarrow$ big i so ind's frequ. dependent res. nature is small *inductive reactance*—frequency dep. resistive nature of in an RL circuit: $X_L = 2\pi\nu L$ ohms also, "resistor-like" resistance	low pass	with minimal resistance-like resistance in circuit, voltage LEADS current by $\frac{\pi}{2}$ radians
RLC ckt.	Z				

Intro to Capacitors

- **Capacitors**:
  - Capacitors do NOT have a “resistor-like” resistive nature to them. They do have a resistive nature, though, that is depends upon the frequency being impressed upon them by the AC source.
  - This resistive nature, called the capacitive reactance, has the units of ohms (it DOES identify resistance to current flow) and, as has been said above, is frequency dependent.
  - To see why a capacitor has this a frequency-dependent resistive nature, consider the RC circuit to the right.



Conceptually:

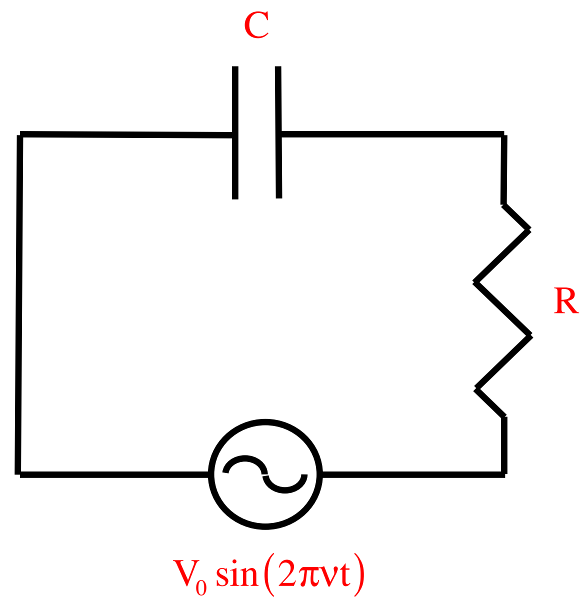


The charge q on a capacitor plate is proportional to the voltage V across the plates (q = CV), so for low frequency when there's charge on the plates for long periods of time, the voltage across the cap is large and the voltage across the resistor will be small . . . which means low or no current in the circuit.

At high frequency when the average charge on the plates over time will be small if not zero, the voltage across the cap will be small, the voltage across the circuit's resistor will be large and so will be the current.

Consequence: Capacitors pass high frequency signal and wipe out low frequency signals (hence the moniker "high pass filter."

The conceptually rationale for capacitive reactance:



1.) From the definition of capacitance, $q = CV$. As $i = dq/dt$, we can write:

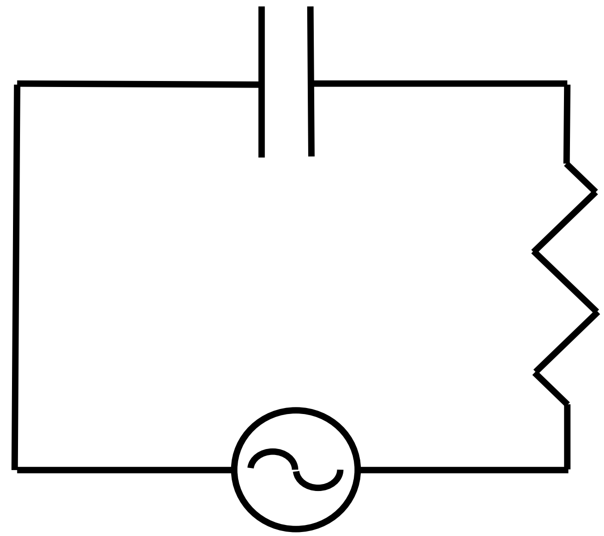
$$\frac{dq}{dt} = i = \frac{1}{C}\frac{dV}{dt}$$

2.) So the current in the circuit is proportional to the rate at which the voltage changes.

3.) At low frequency, the voltage in the circuit changes very slowly, so the current will be low suggesting that the resistive nature (the capacitive reactance) must be large. The opposite happens at high frequency.

4.) Because caps wipe out low frequency and pass high frequency, capacitors are called high pass filters.

Quantitatively, what is the capacitor's resistive nature?



a.) Once again, Kirchoff's loop equation for the circuit yields:

$$-\frac{q}{C} - iR + V_o \sin(2\pi\nu t) = 0$$

b.) Substituting $i = dq/dt$ and rearranging yields:

$$-\frac{dq}{dt}R - \frac{q}{RC} + \frac{V_o}{R}\sin(2\pi\nu t) = 0$$

c.) As before, the solution of this differential equation will be of the form:

$$i = \frac{\text{voltage term}}{\text{net resistive nature}}$$

d.) If we did the math, part of that denominator would be a frequency-dependent quantity

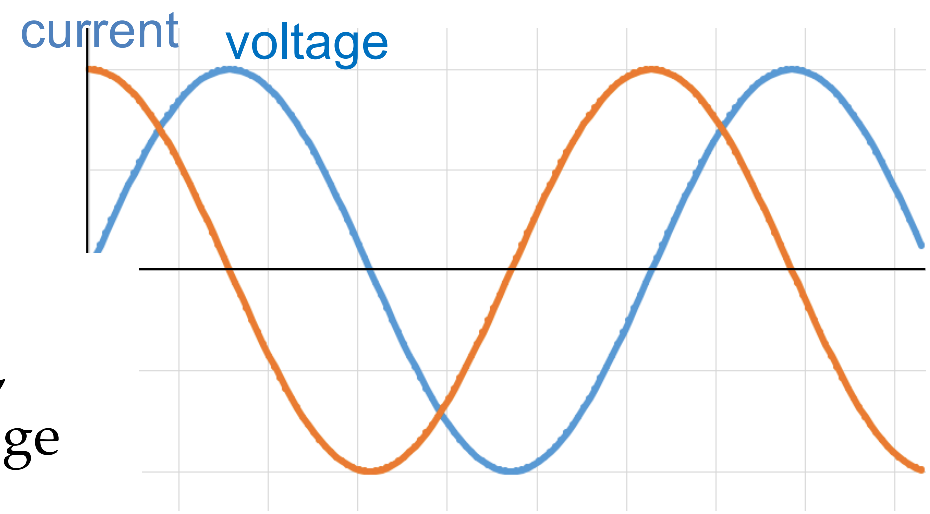
$$\frac{1}{2\pi\nu C} \text{ (ohms)}$$

e.) This is the capacitive reactance X_C, the frequency-dependent resistive nature of a capacitor.

What’s the deal with the phase shift?

1.) We’ve already established that:

$$i = \frac{1}{C}\frac{dV}{dt}$$

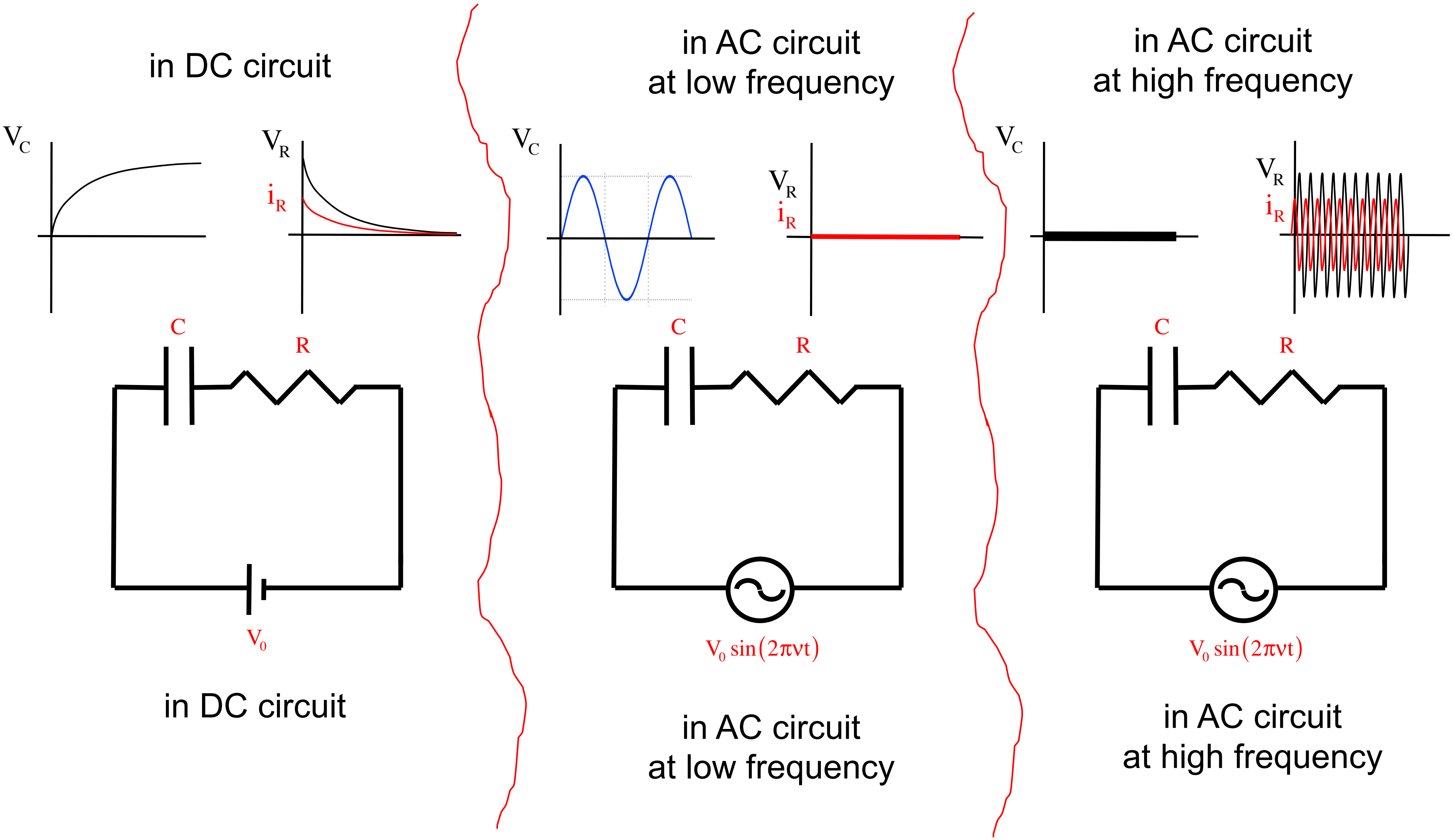


2.) So the derivative of the voltage function yields the graph to the right, and from it you can see that the voltage lags the current.

3.) In fact, for a circuit with minimal resistor-like resistance in it, that lag will be $\pi/2$ radians .

4.) Conceptually, this makes sense: when the capacitor is empty (V_{cap} ~ 0), current will flow to charge it (big I, low V). When the capacitor is charged, there’s no longer current flowing (no I, max V). So voltage lags behind the current.

RC circuit:


in DC circuit
V_C
V_R
i_R
C
R
V_0
in DC circuit
in AC circuit
at low frequency
V_C
V_R
i_R
C
R
V_0 sin(2πνt)
in AC circuit
at low frequency
in AC circuit
at high frequency
V_C
V_R
i_R
C
R
V_0 sin(2πνt)
in AC circuit
at high frequency

element	symbol	units	resistive nature	Filter?	phase relationship
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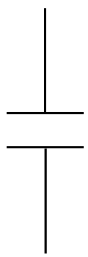
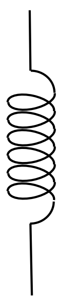
Capacitor	 C	farads	at low ν, V_c big, V_R low $\Rightarrow$ low i so cap's frequ. dependent res. nature is big *capacitive reactance*—frequency dep. resistive nature of in an RC circuit: $X_C = \frac{1}{2\pi\nu C}$ ohms	high pass	with minimal resistance-like resistance in circuit, voltage LAGS current by $\frac{\pi}{2}$ radians
Inductor	 L	henrys	at low ν, V_L small, V_R big $\Rightarrow$ big i so ind's frequ. dependent res. nature is small *inductive reactance*—frequency dep. resistive nature of in an RL circuit: $X_L = 2\pi\nu L$ ohms also, "resistor-like" resistance	low pass	with minimal resistance-like resistance in circuit, voltage LEADS current by $\frac{\pi}{2}$ radians
RLC ckt.					

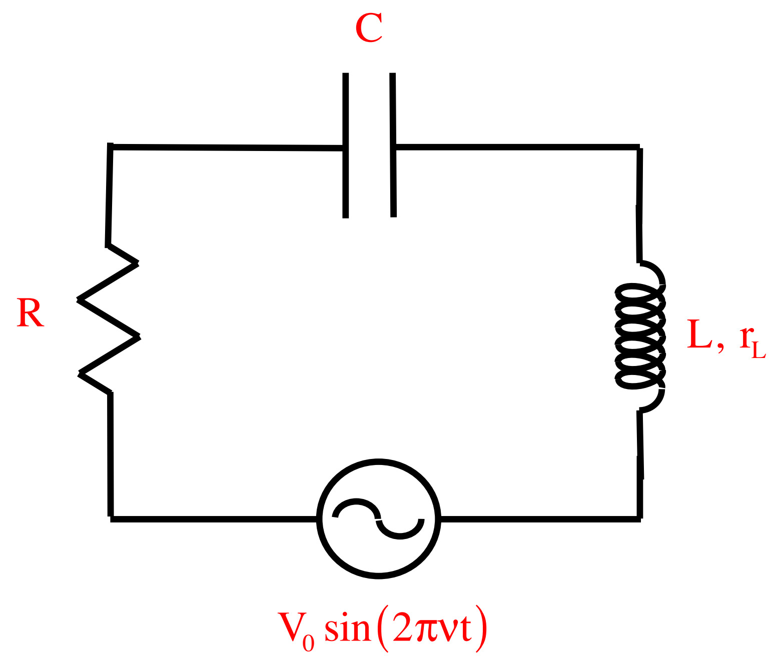
RLC circuits - finally!

So what happens when we have a resistor, capacitor, and inductor all in the same circuit (and RLC circuit)?

All hell breaks loose:

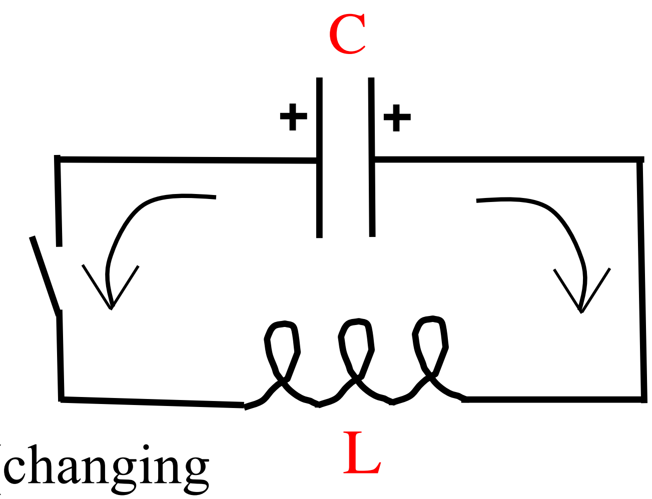
The inductor will try to make the voltage *lag* the current and fights to suppress the AC source signal if it happens to be high frequency;

The capacitor will try to make the voltage *lead* the current and fights to suppress the AC source signal if it happens to be low frequency;



So you'd expect there would be NO frequency that would produce current in the circuit . . . except that's not the case. There will be one frequency at which the lead/lag characteristics of the inductor and capacitor will nullify one another leaving only the resistor-like resistance in the circuit to limit current.

But before we look at the math, consider the following situation conceptually:



1.) Consider a charged capacitor and an inductor in a circuit with very little resistor-like resistance.

2.) When the switch is thrown, the capacitor begins to discharge, attempting to send current through the inductor;

3.) The inductor responds by producing a back EMF (changing magnetic flux, after all) that fights the increase in charge flow.

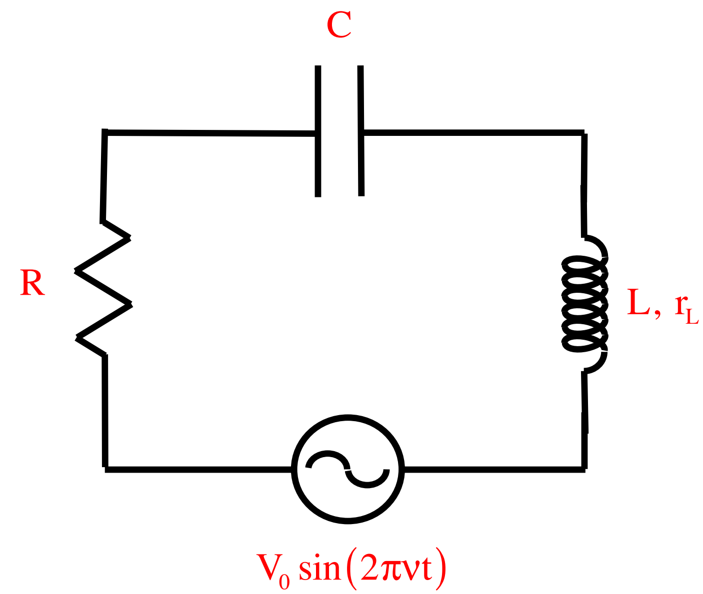
4.) As the cap's charge diminishes and the current slows, the inductor again fights the change by making charge flow MORE than it would have; by the time the cap runs out of charge the inductor will still be forcing current to flow.

5.) This will begin to charge up the other plate.

6.) Once the recharging stops, the cap will begin to discharge going the other way, and the cycle will replay itself. With no resistance, this will continue indefinitely.

7.) Bottom line: Once the process starts, current will oscillate in the circuit at some characteristic frequency, called the resonance frequency; that frequency will be dependent upon the size of the cap and inductor in the circuit.

There are two ways to go about looking at this. The first has to do with what are called phasor diagrams.



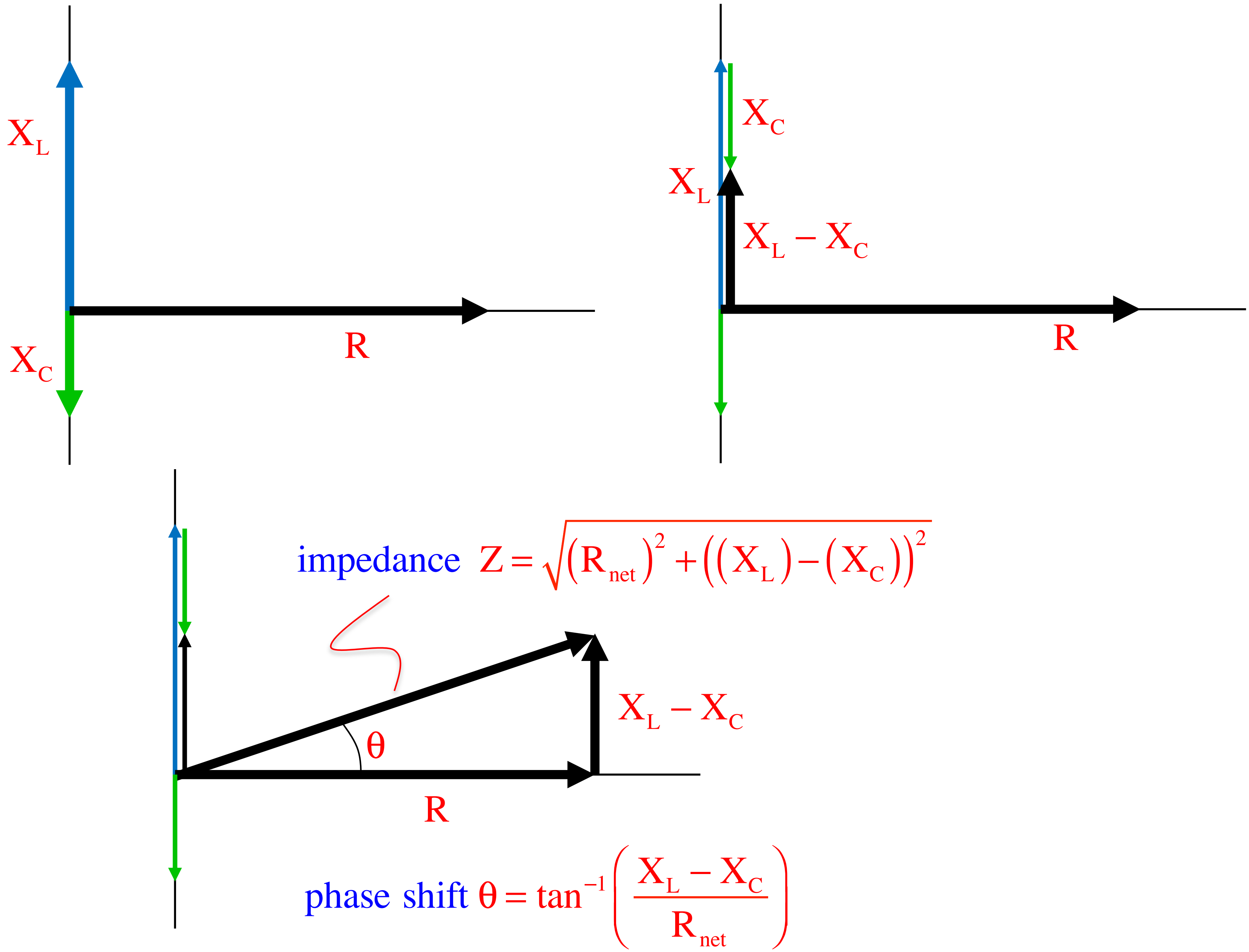
--In a phasor diagram, the resistive nature of all of the elements that don't throw the voltage out of phase with the current are graphed along the *x axis*;

--The resistive nature of all of the elements that make the voltage lead the current by a quarter of a cycle are graphed along the +*y axis*;

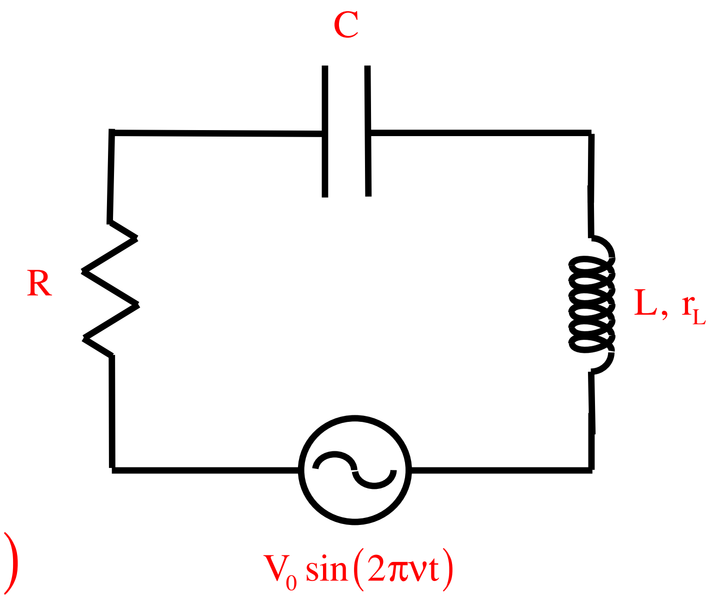
--The resistive nature of all of the elements that make the voltage lag the current by a quarter of a cycle are graphed along the -*y axis*;

--Once done, the vectors are added . . . (which is all happening on the next page)

Phasor diagram for an RLC circuit



To the math: Kirchoff's loop equation yields:



$$-L\frac{di}{dt} + V_o \sin(2\pi\nu t) - i(R + r_L) - \frac{q}{C} = 0$$

$$\Rightarrow \quad \frac{di}{dt} + \frac{(R + r_L)}{L} i + \frac{q}{LC} = \frac{V_o}{L} \sin(2\pi\nu t)$$

$$\Rightarrow \quad \frac{dq^2}{d^2t} + \frac{(R + r_L)}{L}\frac{dq}{dt} + \left(\frac{1}{LC}\right) q = \frac{V_o}{L} \sin(2\pi\nu t)$$

5.) Solving this second-order differential equation in q (which is a trip unto itself) will produce a solution for i whose denominator looks like:

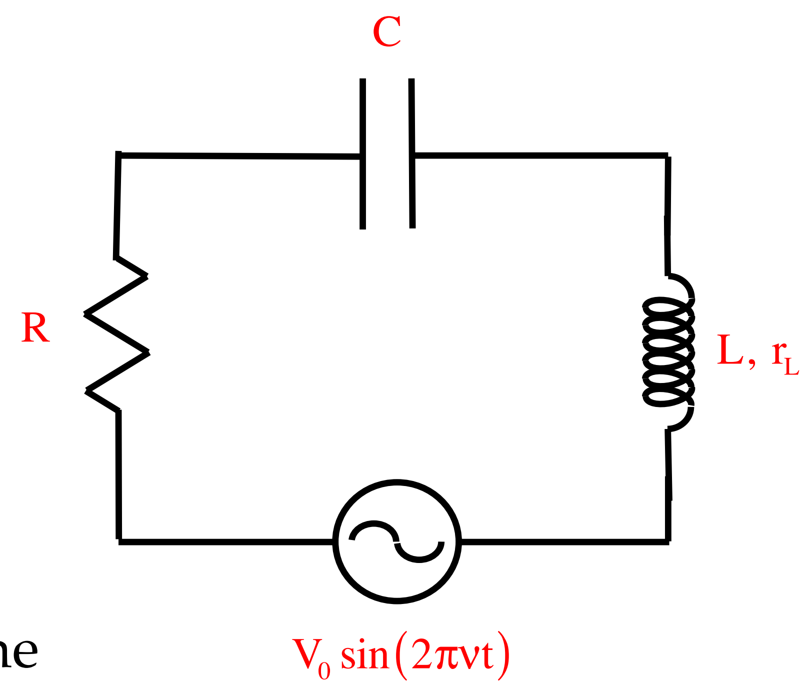
$$i = \frac{\text{voltage term}}{\text{net resistive nature}}$$

$$\left[(R + r_L)^2 + \left(2\pi\nu L - \frac{1}{2\pi\nu C}\right)^2\right]^{1/2}$$

6.) This overall "frequency dependent" resistive nature for the RLC circuit, which is sometimes written as $Z = \left[(R_{net})^2 + \left((X_L)^2 - (X_C)^2\right)\right]^{1/2}$,

is called the circuit's impedance and is given (as shown) the symbol Z.

7.) Things to notice about this relationship.



a.) Ohm's Law works just fine in these circuits, except now it is written as $V = i\,Z$, where the impedance Z is the net resistive nature of the circuit.

b.) With no capacitor or inductor in the circuit, the impedance simply becomes the resistance R in the circuit. That means $V = iR$ Is really just a special case of this more expanded version.

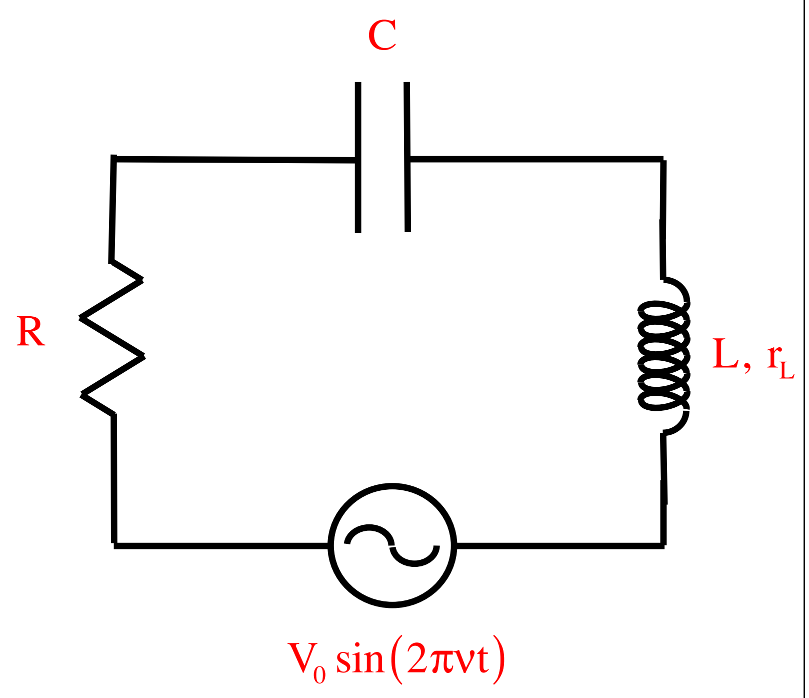
c.) There is a frequency at which the impedance is a minimum and the current is a maximum. It happens when the $\left(2\pi\nu L - \frac{1}{2\pi\nu C}\right)$ part goes to zero.

That is, when $2\pi\nu L = \frac{1}{2\pi\nu C}$, or when:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

This is that resonance frequency at which the circuit's current naturally wants to oscillate.

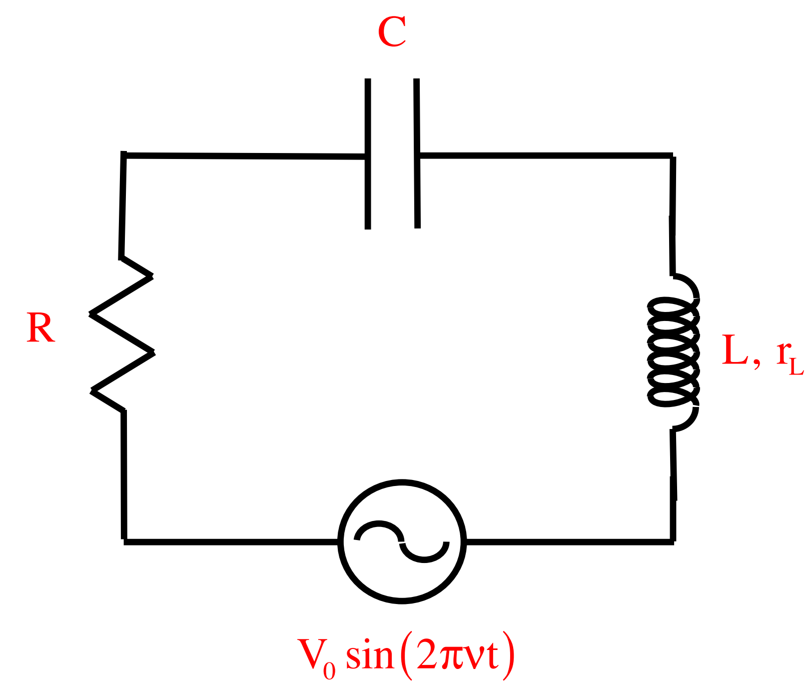
d.) At resonance frequency, the only resistance in the circuit is resistor-like resistance and that signal proliferates.



i.) What this means is that the frequency response curve for a typical RLC circuit looks like:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

e.) This frequency-response characteristic is going to be very important in the radio circuit (more about this later).



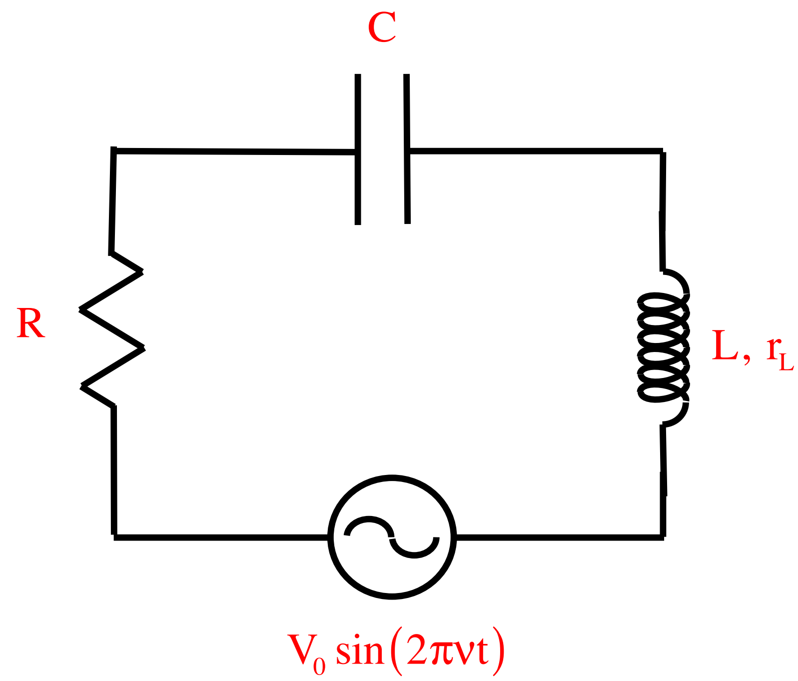
f.) And as an interesting note: you now know what the impedance label on the back of your stereo speakers means. It is telling the you speaker's net, overall resistance to charge flow due to all of the resistors, capacitors or inductors in the speaker circuit when the frequency is between 250 Hz and 400 Hz.

In other words, when a speaker says, Impedance: 8 ohms, it means that for the circuit when run at, say, 300 Hz,

$$Z = \left[\left(R + r_L\right)^2 + \left(2\pi\nu L - \frac{1}{2\pi\nu C}\right)^2 \right]^{1/2} = 8\ \Omega.$$

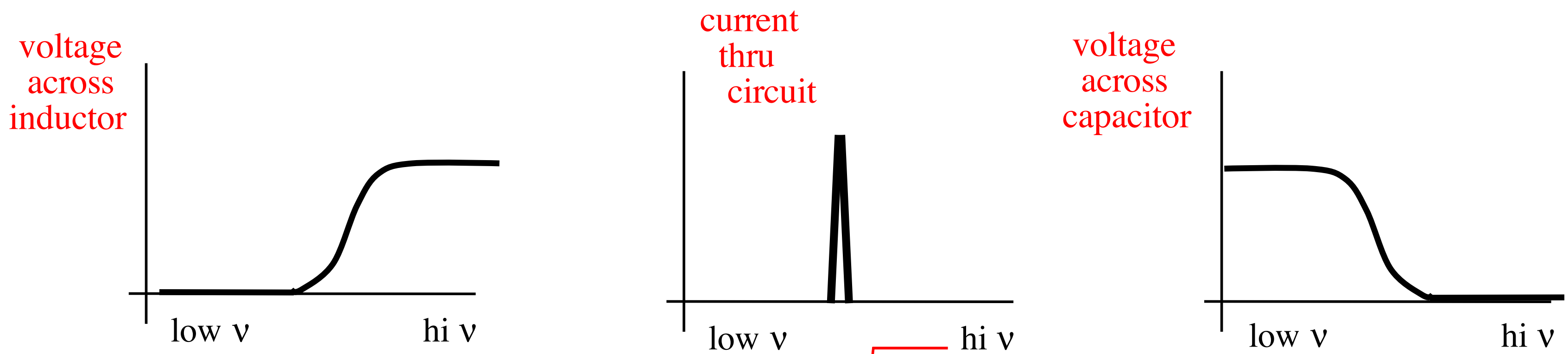
8.) Lastly, it should be noted that the calculation of the phase shift

$$\theta = \tan^{-1}\left(\frac{X_L - X_C}{R_{net}}\right)$$

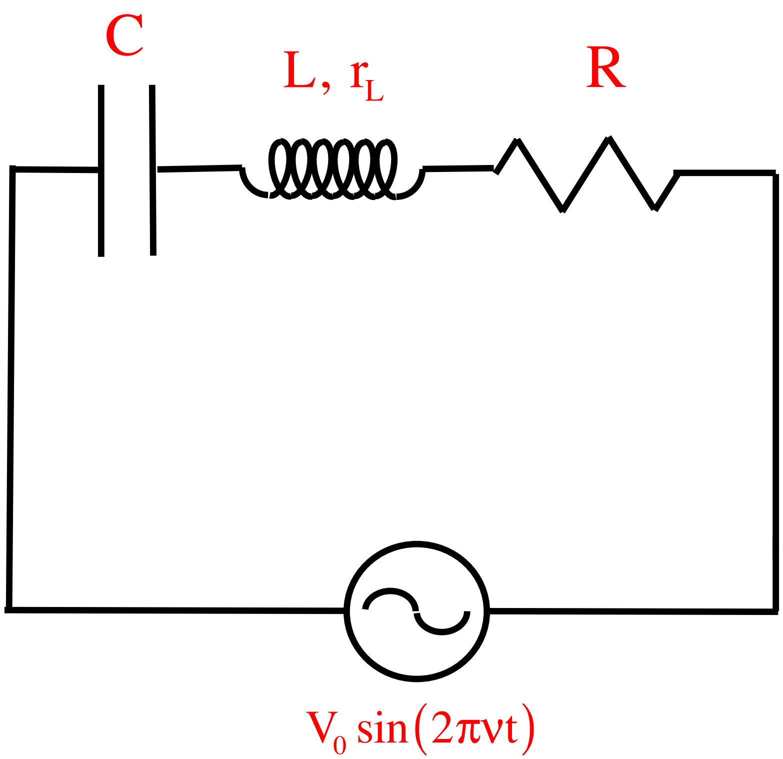


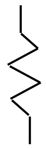
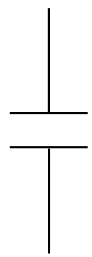
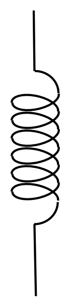
--whether the voltage leads or lags the current--identifies which is happening by its sign. If the angle is positive, it means the voltage leads the current; if the angle is negative, it means the voltage lags the current.

RLC circuit: As we pan through the frequencies:



$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$



element	symbol	units	resistive nature	Filter?	phase relationship
Resistor	 R	ohms	"resistor-like" resistance R (ohms)	no	current in phase with voltage across element
Capacitor	 C	farads	at low ν, V_c big, V_R low $\Rightarrow$ low i so cap's frequ. dependent res. nature is big *capacitive reactance*—frequency dep. resistive nature of in an RC circuit: $$X_C = \frac{1}{2\pi\nu C} \text{ ohms}$$	high pass	with minimal resistance-like resistance in circuit, voltage LAGS current by $\frac{\pi}{2}$ radians
Inductor	 L	henrys	at low ν, V_L small, V_R big $\Rightarrow$ big i so ind's frequ. dependent res. nature is small *inductive reactance*—frequency dep. resistive nature of in an RL circuit: $$X_L = 2\pi\nu L \text{ ohms}$$ also, "resistor-like" resistance r_L	low pass	with minimal resistance-like resistance in circuit, voltage LEADS current by $\frac{\pi}{2}$ radians
RLC ckt.	Z	ohms	***Impedance = total resistive nature of circuit*** $$Z = \left[(R + r_L)^2 + \left(2\pi\nu L - \frac{1}{2\pi\nu C} \right)^2 \right]^{1/2}$$ Minimum Z, max i at resonance frequ. when $\nu_{resonance} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$	passes res. frequ.	phase shift defined by: $$\theta = \tan^{-1}\left(\frac{X_L - X_C}{R_{net}} \right)$$ V leads i if $\theta > 0$, V leads i if $\theta < 0$, V lags i